

# Projected p-wave superconducting wave-functions for topological orders

Su-Peng Kou<sup>1,\*</sup> and Xiao-Gang Wen<sup>2</sup>

<sup>1</sup>*Department of Physics, Beijing Normal University, Beijing 100875, China*

<sup>2</sup>*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*

In this paper we develop a systematic theory for topological orders by the projected  $p$ -wave superconducting (SC) wave-functions and unify the different topological orders for spin models into the fermionic picture. We found that the energy for the fermions at  $\mathbf{k} = (0,0), (0,\pi), (\pi,0), (\pi,\pi)$  acts as a topological invariant to characterize 16 universal classes of different topological orders for the spin models with translation invariance. Based on the projected  $p$ -wave SC wave-functions the topological properties for the known topological orders in the exact solved spin models are obtained. Finally new types of topological orders are predicted.

Keywords: topological order,  $p$ -wave superconductivity, topological degeneracy

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Topological order is a new paradigm that lies beyond the realm of Landau's theory based on the concept of symmetry-breaking for classical statistical systems. Physically, the topological orders describe the quantum orders with the long range entanglements in a gapped quantum ground state. A well-known example for the topological order is the fractional quantum Hall (FQH) state, which carries fractional charge and anyon statistics. To learn the nature for the FQH states, Laughlin wave function is a good starting point. Recently, many spin models with different topologically ordered states were found, such as the toric-code model [1], the Wen's plaquette model [2, 3] and the Kitaev model on a hexagonal lattice [4]. It is interesting that all the topological orders of the exact solved spin models can be described by  $p$ -wave superconducting (SC) wave functions [3, 5, 6, 7, 8]. People may ask two questions: if the topological orders for the different spin models are the same one? and how to use  $p$ -wave SC wave functions to describe the topological properties of the spin models? In this paper, we would like to develop a systematic theory for the topological ordered states based on the (projected)  $p$ -wave SC wave-functions and learn the nature of the different kinds of topological orders for the spin models.

*The general "mean-field" spinless fermion Hamiltonian* - We start with the general "mean-field" spinless fermion Hamiltonian with nonzero  $p$ -wave SC order parameter

$$H_{mean} = \sum_{ij} \psi_i^\dagger u_{ij} \psi_j + \sum_{ij} (\psi_i^\dagger \eta_{ij} \psi_j^\dagger + h.c.) + \mu \sum_i \psi_i^\dagger \psi_i. \quad (1)$$

where  $u_{ij}$ ,  $\eta_{ij}$  are 2 by 2 complex matrices.  $\mu$  is the chemical potential. Let  $|\Psi_{mean}^{(u_{ij}, \eta_{ij})}\rangle$  be the ground state of  $H_{mean}$ . To represent the topological order with full gapped excitations, each site must have just single fermion. Then we add a constraint to the  $p$ -wave SC wave functions. The topological ordered state can be obtained from the mean-field SC state  $|\Psi_{mean}^{(u_{ij}, \eta_{ij})}\rangle$  by

projection it into the subspace with single fermion per site,  $|\Psi_{spin}^{(u_{ij}, \eta_{ij})}\rangle = P|\Psi_{mean}^{(u_{ij}, \eta_{ij})}\rangle$ . Here one has the projection operator as  $P = \prod_i \frac{1 - (-1)^{\psi_i^\dagger \psi_i}}{2}$ . Because the fermion number can only be 0 or 1 on each site, the constraint for the fermion number on each site can be reduced into a constraint for total fermion number! Because the fermion number for the superconducting state is not conserved, we can only distinguish the ground state with an even total fermion number or an odd one. On the even-by-even (e\*e), even-by-odd (e\*o) and odd-by-even (o\*e) lattices, the total fermion number  $N$  must be even, the projection operator turns into  $P = P_r^e = \frac{1 + (-1)^{\hat{N}}}{2}$ . On an odd-by-odd (o\*o) lattice, the total fermion number must be odd, the projection operator  $P$  is reduced into  $P_r^o = \frac{1 - (-1)^{\hat{N}}}{2}$ . Here  $\hat{N} = \sum_i \psi_i^\dagger \psi_i = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger \psi_{\mathbf{k}}$  is the total fermion number operator.

For the translation invariant ansatz, one can rewrite the above "mean-field" fermion Hamiltonian in momentum space as  $H_{mean} = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger U(\mathbf{k}) \Psi_{\mathbf{k}}$  with  $\Psi_{\mathbf{k}} = \begin{pmatrix} \psi_{\mathbf{k}} \\ \psi_{-\mathbf{k}}^\dagger \end{pmatrix}$  and  $\Psi_{\mathbf{k}}^\dagger = (\psi_{\mathbf{k}}^\dagger \ \psi_{-\mathbf{k}})$ . We note that  $(\psi_{-\mathbf{k}}^\dagger, \psi_{-\mathbf{k}})$  can be expressed in terms of  $(\psi_{\mathbf{k}}^\dagger, \psi_{\mathbf{k}})$  from  $\Psi_{-\mathbf{k}} = \Gamma \Psi_{\mathbf{k}}^*$  and  $\Psi_{-\mathbf{k}}^\dagger = \Psi_{\mathbf{k}}^T \Gamma$  with  $\Gamma = \sigma_1$ . Then we get that  $U(\mathbf{k})$  has an additional constraint  $U(\mathbf{k}) = -\Gamma U^T(\mathbf{k}) \Gamma$ . Under the constraint, we divide 4  $2 \times 2$  matrices  $M_\alpha = (\mathbf{1}, \sigma)$  into two classes: in one class, the matrix  $M_3 = \sigma_3$  commutes with  $\Gamma$ , we call it "even matrix"; the others  $M_0 = \mathbf{1}$ ,  $M_1 = \sigma_1$  and  $M_2 = \sigma_2$  anti-commute with  $\Gamma$ , we call it "odd matrices". Then one can expand  $U(\mathbf{k})$  as  $U(\mathbf{k}) = \sum_{\alpha} u_\alpha(\mathbf{k}) M_\alpha$ ,  $\alpha = 0, 1, 2, 3$ . We can prove that  $u_3(\mathbf{k}) = u_3(-\mathbf{k})$  and  $u_\alpha(\mathbf{k}) = -u_\alpha(-\mathbf{k})$ ,  $\alpha = (0, 1, 2)$ . Thus  $u_\alpha(\mathbf{k})$  ( $\alpha = (0, 1, 2)$ ) are odd functions of  $k_x, k_y$  and are fixed to be zero at momenta  $(0,0)$ ,  $(0,\pi)$ ,  $(\pi,0)$ ,  $(\pi,\pi)$ :  $u_\alpha(\mathbf{k} = 0) \equiv 0$ . Here  $\mathbf{k} = 0$  denote four special points in momentum space  $(0,0)$ ,  $(0,\pi)$ ,  $(\pi,0)$ ,  $(\pi,\pi)$ .

In general,  $U(\mathbf{k})$  is written into

$$U(\mathbf{k}) = \begin{pmatrix} \varepsilon_{\mathbf{k}} & \Delta_{1,\mathbf{k}} + i\Delta_{2,\mathbf{k}} \\ \Delta_{1,\mathbf{k}} - i\Delta_{2,\mathbf{k}} & -\varepsilon_{\mathbf{k}} \end{pmatrix}$$

where  $\varepsilon_{\mathbf{k}}$  is the even function of  $k_x$  and  $k_y$  and  $\Delta_{1,\mathbf{k}}, \Delta_{2,\mathbf{k}}$  are the odd functions of  $k_x$  and  $k_y$ . At the points  $\mathbf{k} > 0$ , one has

$$H(\mathbf{k} > 0)_{\text{mean}} = \sum_{\mathbf{k} > 0} \begin{pmatrix} \alpha_{\mathbf{k}}^\dagger & \beta_{\mathbf{k}}^\dagger \end{pmatrix} \begin{pmatrix} \varepsilon(\mathbf{k}) & 0 \\ 0 & -\varepsilon(\mathbf{k}) \end{pmatrix} \begin{pmatrix} \alpha_{\mathbf{k}} \\ \beta_{\mathbf{k}} \end{pmatrix}$$

where  $\alpha_{\mathbf{k} > 0}$  and  $\beta_{\mathbf{k} > 0} = \alpha_{-\mathbf{k}}^\dagger$  are (diagonalized) quasi-particle operators. Here  $\mathbf{k} > 0$  means that  $k_y > 0$  or  $k_y = 0, k_x > 0$ . At the four points  $(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$ , the “mean-field” Hamiltonian is diagonalized into

$$H(\mathbf{k} = 0)_{\text{mean}} = \sum_{\mathbf{k}=0} \varepsilon(\mathbf{k}) \psi_{\mathbf{k}}^\dagger \psi_{\mathbf{k}}.$$

The ground state is denoted by a projected  $p$ -wave BCS type wave-function as

$$|g\rangle = |\Omega\rangle = P \prod_{\mathbf{k}} |u_{\mathbf{k}}|^{1/2} \exp\left(\frac{1}{2} \sum_{\mathbf{k}} g_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger \psi_{-\mathbf{k}}^\dagger\right) |0\rangle, \quad (2)$$

where  $g_{\mathbf{k}} = v_{\mathbf{k}}/u_{\mathbf{k}}$  and  $v_{\mathbf{k}}^2 = \frac{1}{2} \left(1 - \frac{\varepsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$ ,  $u_{\mathbf{k}}^2 = \frac{1}{2} \left(1 + \frac{\varepsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$ . Here  $E_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + (\Delta_{1,\mathbf{k}})^2 + (\Delta_{2,\mathbf{k}})^2}$  is the energy dispersion.  $P$  is the projected operator which guarantees a constraint to the Hilbert space with single fermion on each site. For the fermionic vacuum we have  $\psi_{\mathbf{k}}|0\rangle = 0$  and all the negative energy levels are filled by the fermions. Under these conditions, the ground state of Eq.(2) is not a  $p$ -wave superconducting state, instead it is an insulator of topological order!

In the following part we will use the projected  $p$ -wave SC wave functions to learn the nature of the topological orders, including the classification of topological orders for spin models with translation invariance, the degeneracy of the ground states and the prediction of new types of topological orders.

*Classification of topological orders* - Firstly, we classify the topological orders on the lattice by the projected  $p$ -wave SC wave-functions. Because there is no local order parameter to distinguish different topological phases, we classify the topological phases by the quantum phase transitions between them. Since the quantum states with topological order are protected by the finite energy gap for excitations and stable against any local perturbations, to break down a topological phase, one needs to close the energy gap for the excitations[9, 10]. That is the *unmovable gapless excitations* indicate a quantum phase transition between different topological orders.

For the topological orders described by the projected  $p$ -wave SC wave-functions, we find that  $\varepsilon(\mathbf{k} = 0) = 0$  at one or more points of  $\mathbf{k} = 0$  define the quantum

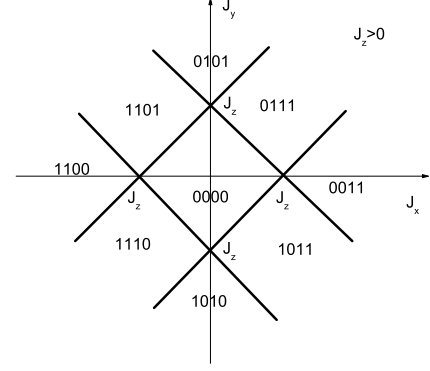


FIG. 1: Classification of topological orders by a  $p$ -wave SC wave function for  $J_z > 0$ .

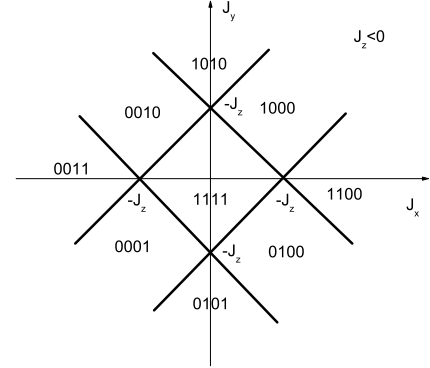


FIG. 2: Classification of topological orders by a  $p$ -wave SC wave function for  $J_z < 0$ .

topological transitions between different topological orders. For example, the topological order describe by the projected  $p$ -wave SC wave-function with  $\varepsilon(\mathbf{k} = 0) > 0$  and that by the projected  $p$ -wave SC wave-function with  $\varepsilon(\mathbf{k} = 0) < 0$  cannot be the same one. The energy gap must vanish at  $\mathbf{k} = 0$ , corresponding to the gapless critical states separating two topological phases. Then the signs of  $\varepsilon(\mathbf{k} = 0)$  become an additional structure beyond projected spin group to distinguish topological phases. There are totally 16 different cases for the signs of  $\varepsilon(\mathbf{k} = 0)$  which characterize 16 different universal classes of topological orders. We use the symbols “ $\mu\nu\lambda\xi$ ” to denote them.  $\mu$ (or  $\nu, \lambda, \xi$ ) = 1 denotes the case for  $\varepsilon(\mathbf{k} = (0, 0)) < 0$  (or  $\varepsilon(\mathbf{k} = (\pi, 0)) < 0, \varepsilon(\mathbf{k} = (0, \pi)) < 0, \varepsilon(\mathbf{k} = (\pi, \pi)) < 0$ ) and  $\mu$ (or  $\nu, \lambda, \xi$ ) = 0 denotes the case  $\varepsilon(\mathbf{k} = (0, 0)) > 0$  (or  $\varepsilon(\mathbf{k} = (\pi, 0)) < 0, \varepsilon(\mathbf{k} = (0, \pi)) < 0, \varepsilon(\mathbf{k} = (\pi, \pi)) < 0$ ).

*Fermionic wave-functions for the topological orders* - let’s give one example for each class of topological order by the  $p$ -wave SC wave-functions. A general mean field

$p$ -wave SC wave-function is written as

$$U(\mathbf{k}) = \begin{pmatrix} \varepsilon_{\mathbf{k}} & \Delta_{1,\mathbf{k}} + i\Delta_{2,\mathbf{k}} \\ \Delta_{1,\mathbf{k}} - i\Delta_{2,\mathbf{k}} & -\varepsilon_{\mathbf{k}} \end{pmatrix},$$

$$\varepsilon_{\mathbf{k}} = J_z - J_x \cos k_x - J_y \cos k_y,$$

$$\Delta_{1,q} = J \sin k_x + J \sin k_y, \Delta_{2,\mathbf{k}} = 0, \quad (3)$$

with  $J \neq 0$ . The classification of the topological orders for above wave-function is shown in the fig.1 and fig.2. From the two figures, one can see that the topological orders of class 1001 and class 0110 cannot be denoted by above wave functions. For 1001, instead, a given ansatz is  $\varepsilon_{\mathbf{k}} = -J \cos(k_x + k_y)$ ,  $\Delta_{1,\mathbf{k}} = J' \sin k_x$ ,  $\Delta_{2,\mathbf{k}} = -J \sin(k_x + k_y)$ . For 0110, a given ansatz is  $\varepsilon_{\mathbf{k}} = J \cos(k_x + k_y)$ ,  $\Delta_{1,\mathbf{k}} = J' \sin k_x$ ,  $\Delta_{2,\mathbf{k}} = J \sin(k_x + k_y)$ .

*Topological degeneracies* - We calculate the topological degeneracies for different classes of topological orders by using the  $p$ -wave SC wave-functions. Now we assume  $(u_{ij}^{(m,n)}, \eta_{ij}^{(m,n)}) = ((-)^{ms_x(ij)}(-)^{ns_y(ij)}\bar{u}_{ij}, (-)^{ms_x(ij)}(-)^{ns_y(ij)}\bar{\eta}_{ij})$  to note four degenerate ground states for the topological orders on a torus. Take  $m, n = 0, 1$  as an example.  $s_{x,y}(ij)$  have values 0 or 1, with  $s_{x,y}(ij) = 1$  if the link  $ij$  crosses the  $x$  or  $y$  line and  $s_{x,y}(ij) = 0$  otherwise. However, some ansatzs may be not permitted by the projections  $P_r^e$  (or  $P_r^o$ ). The ground state  $|\Psi_{\text{mean}}^{(u_{ij}^{(m,n)}, \eta_{ij}^{(m,n)})}\rangle$  with even (or odd) number particles is only allowed under projection  $P_r^e$  (or  $P_r^o$ ),

$$|\Psi_{\text{spin}}\rangle = P_r^{e,o} |\Psi_{\text{mean}}^{(u_{ij}^{(m,n)}, \eta_{ij}^{(m,n)})}\rangle = |\Psi_{\text{mean}}^{(u_{ij}^{(m,n)}, \eta_{ij}^{(m,n)})}\rangle. \quad (4)$$

To obtain the topological degeneracy, one needs to deal with the projection operators  $P_r^e$  and  $P_r^o$  carefully. Let's simplify the projection operators firstly. We note that, for  $\mathbf{k} > 0$ ,

$$N_{\mathbf{k}>0} = \alpha_{\mathbf{k}>0}^\dagger \alpha_{\mathbf{k}>0} - \beta_{\mathbf{k}>0}^\dagger \beta_{\mathbf{k}>0} + 1. \quad (5)$$

For the mean-field ansatz defined above, there always exists a negative energy level for fermions at the point  $\mathbf{k} > 0$ . For the energy of  $\beta$  band  $-\varepsilon(\mathbf{k})$  is negative,  $\varepsilon(\mathbf{k}) > 0$ ,  $\beta$  band is occupied and  $\alpha$  band is empty, we have  $N_{\mathbf{k}>0} = 0$ . On the other hand, for the energy of  $\alpha$  band  $\varepsilon(\mathbf{k})$  is negative,  $\varepsilon(\mathbf{k}) < 0$ ,  $\alpha$  band is occupied and  $\beta$  band is empty, we have  $N_{\mathbf{k}>0} = 2$ . As a result, the total number of  $\psi$  fermions on every point of  $\mathbf{k} > 0$ ,  $\sum_{\mathbf{k}>0} N_{\mathbf{k}}$ , is even. Then  $(-1)^{\hat{N}}$  is reduced into  $(-1)^{\hat{N}_0}$  with  $\hat{N} = \sum_{\mathbf{k}>0} N_{\mathbf{k}} + \hat{N}_0$ . Here  $\hat{N}_0 = \sum_{\mathbf{k}=0} \psi_{\mathbf{k}}^\dagger \psi_{\mathbf{k}}$  is the total fermion number on the four special points  $\mathbf{k} = 0$ .

So to determine if the mean-field ground state contain even or odd number of  $\psi$  fermions, we only need to examine the occupation on the four special points:  $(0,0)$ ,  $(0,\pi)$ ,  $(\pi,0)$ ,  $(\pi,\pi)$  and calculate the total fermion number on them. The projection operators now are reduced into  $P_r^e = \frac{1+(-1)^{\hat{N}_0}}{2}$  and  $P_r^o = \frac{1-(-1)^{\hat{N}_0}}{2}$ . For a given point  $\mathbf{k} = 0$ , the energy for  $\psi$  is  $\varepsilon(\mathbf{k} = 0)$ . Then if  $\varepsilon(\mathbf{k} = 0) < 0$ , one  $\psi$  particle occupies energy level at the point  $(0,0)$  (or  $(0,\pi)$ ,  $(\pi,0)$ ,  $(\pi,\pi)$ ); if  $\varepsilon(\mathbf{k} = 0) > 0$ , zero  $\psi$  particle occupies the energy level at  $(0,0)$  (or  $(0,\pi)$ ,  $(\pi,0)$ ,  $(\pi,\pi)$ ). Finally we obtain the projection operators as

$$\frac{1 \pm (-1)^{\hat{N}_0}}{2} = \frac{1 \pm (-1)^{\sum_{\mathbf{k}=0} \zeta(\varepsilon(\mathbf{k}))}}{2}$$

where the function  $\zeta(x)$  denotes  $\zeta(x) = 1$ , for  $x < 0$  and  $\zeta(x) = 0$ , for  $x > 0$ . One can check the occupation number at  $\mathbf{k} = 0$  easily to know if a given ansatz is valid. Using this method, we obtain the topological degeneracy on different lattices for the 16 topological orders. The results are given in the following tables in detail :

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	1111	1110	1101	1011	0111	1100	1010	1001	0110	0011	0101	1000	0100	0010	0001	0000
$(e * e)$	4	3	3	3	3	4	4	4	4	4	4	3	3	3	3	4
$(e * o)$	4	3	3	3	3	4	2	2	2	4	2	3	3	3	3	4
$(o * e)$	4	3	3	3	3	2	4	2	2	2	4	3	3	3	3	4
$(o * o)$	-	3	3	3	3	2	2	2	2	2	2	1	1	1	1	4

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From the results, we can see that for the 16 topological orders, there are 8 Z2 topological orders (1111, 1100, 1010, 1001, 0101, 0011, 0110, 0000) and 8 non-Abelian topological orders (1000, 0100, 0010, 0001, 1110, 1101, 1011, 0111).

The Z2 topological orders are described by Z2 projective symmetry groups. Such Z2 topological orders always have 4 degenerate ground states on an even-by-even lattice with periodic boundary condition. And there always exists three types of quasiparticles: Z2 charge, Z2 vortex,

and fermions, respectively[11]. And the fermions can be regarded as bound states of a Z2 charge and a Z2 vortex.

On the other hand, we have two classes of 8 types the non-Abelian topological orders[12]: on an even-by-even lattice the topological degeneracy is always 3. However, on an odd-by-odd lattice, the degeneracy is 3 for first class 4 non-Abelian topological orders and 1 for the other class 4 topological orders. All the 8 non-Abelian topological orders have the two types of quasiparticles: non-Abelian anyons and fermions.

Our results are consistent to those by Read and Green [13]. They have pointed out there are two kinds of  $p_x + ip_y$ -wave fermion paired state: one is in the weak pairing phase as an example of topological order with non-Abelian anyon [13], the other in the strong pairing phase as a topological order with Abelian anyon. However, our results show more rich physics properties for the  $p_x + ip_y$ -wave fermion paired states on lattice beyond the picture of weak and strong pairing phases.

*Unification of the topological orders for exact solved spin models* - Finally we can identify the classes of topological orders for the exact solved spin models by our classification and show their topological properties.

For the Kitaev model on a hexagonal lattice [4], the original spin model is mapped to a model of  $p$ -wave BCS model on a square lattice by a Jordan-Wigner transformation with [5]

$$\begin{aligned}\varepsilon_{\mathbf{k}} &= |J_z| - |J_x| \cos k_x - |J_y| \cos k_y, \\ \Delta_{1,q} &= |J_x| \sin k_x + |J_y| \sin k_y, \Delta_{2,\mathbf{k}} = 0.\end{aligned}\quad (6)$$

From the relationship between  $|J_x|$ ,  $|J_y|$ ,  $|J_z|$ , we found that the phase  $|J_z| > |J_x| + |J_y|$  as 0000 type Z2 topological order;  $|J_y| > |J_x| + |J_z|$  as 1010 type Z2 topological order;  $|J_x| > |J_y| + |J_z|$  as 1100 type Z2 topological order. The ground state degenerate is always 4 on an even-by-even lattice (on a torus). However, on other lattices (even-by-odd, odd-by-even, odd-by-odd), the ground state degenerate is different: for 0000 type, it is 4; for 1010; it is 4 on an odd-by-even lattice, but 2 on an even-by-odd and odd-by-odd lattice; for 1100; it is 4 on an even-by-odd lattice, but 2 on an odd-by-even and odd-by-odd lattice.

For the Kitaev model on a honeycomb lattice with minimal three- and four-spin terms and a T-symmetry breaking external magnetic field, the corresponding  $p$ -wave SC state is given as [6, 7, 8]

$$\begin{aligned}\varepsilon_{\mathbf{k}} &= |J_z| - |\tilde{J}_x| \cos k_x - |\tilde{J}_y| \cos k_y, \\ \Delta_{1,\mathbf{k}} &= \Delta_{1x} \sin k_x + \Delta_{1y} \sin k_y, \\ \Delta_{2,\mathbf{k}} &= \Delta_{2x} \sin k_x + \Delta_{2y} \sin k_y\end{aligned}\quad (7)$$

with  $|J_z| < |\tilde{J}_x| + |\tilde{J}_y|$ ,  $|J_z| > |\tilde{J}_x| - |\tilde{J}_y|$ ,  $|J_z| > -|\tilde{J}_x| + |\tilde{J}_y|$ . It is described by the 1000 type Non-Abelian topological order[4, 14]. The topological degeneracy is 3,

3, 3, 1 for even-by-even, even-by-odd, odd-by-even, odd-by-odd lattices, respectively.

For the Wen's plaquette model, a "mean-field" ansatz  $p$ -wave BCS states becomes[2, 3]

$$H_{mean} = \sum_{\langle ij \rangle} \left( \psi_{I,i}^\dagger u_{ij}^{IJ} \psi_{J,j} + \psi_{I,i}^\dagger \eta_{ij}^{IJ} \psi_{J,j}^\dagger + h.c. \right) \quad (8)$$

where  $I, J = 1, 2$ . The ground state for  $g < 0$  is a topological state with  $-\eta_{i,i+x} = u_{i,i+x} = 1 + \sigma^3$  and  $-\eta_{i,i+y} = u_{i,i+y} = 1 - \sigma^3$ . The quantum wave-function can be regarded as two-flavor spinless projected  $p$ -wave SC. Then the fermion number on each lattice and the total fermion number must be even. Using the same method, one can also calculate the ground state degeneracy which is 4, 2, 2, 2 for even-by-even, even-by-odd, odd-by-even, odd-by-odd lattices, respectively. It is the 0110 type of the topological order.

In addition, it is known that the topological orders of the toric-code model and the Wen-plaquette model are the same ones[1]. So one can use the same  $p$ -wave SC wave-function in Eq.(8) to describe the toric code model.

In conclusion, we develop a systematical theory for topological orders by the (projected)  $p$ -wave SC wave-functions. Based on the theory we unify different topological ordered states for the exact solved spin models and obtain their topological properties. From our classification in this paper, the topological orders in spin models are the same one and at least five different classes of topological orders have been found. That is one can not change the topological state for the Wen-plaquette model into that of the Kitaev model without a quantum phase transition. And we can predict that at the quantum phase transition between them the massless excitations should have zero energy at  $\mathbf{k} = 0$ . In the end, we can also give a prediction that there are totally 11 unknown classes topological orders, including four classes of Z2 topological order ( 1111, 1001, 0011, 0101 ) and 7 classes of non-Abelian topological orders. All the new classes of topological orders need to be explored in the future.

The email for Su-Peng Kou is spkou@bnu.edu.cn. S.P. Kou acknowledges that this research is supported by NFSC Grant no. 10574014.

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\* Electronic address: spkou@bnu.edu.cn

- [1] A. Kitaev, Ann. Phys. **303**, 2(2003). M. H. Freedman, M. Larsen.
- [2] X.-G. Wen, *Quantum Field Theory of Many-Body Systems*, (Oxford Univ. Press, Oxford, 2004).
- [3] X. G. Wen, Phys. Rev. Lett. **90** (2), 016803 (2003).
- [4] A. Kitaev, Ann. Phys. **321**, 2(2006).
- [5] H. D. Chen and J. P. Hu, arXiv:cond-mat/0702366; H. D. Chen and Z. Nussinov, arXiv:cond-mat/0703633.

- [6] Y. Yu and Z. Q. Wang, arXiv:cond-mat/07080631.
- [7] H. Yao, S. A. Kivelson, arXiv: 07080040, Phys. Rev. Lett. to be published.
- [8] S. Yang, D. L. Zhou, C. P. Sun, arXiv:cond-mat/07080040, Phys. Rev. B., to be published.
- [9] X. Y. Feng, G. M. Zhang, and T. Xiang, Phys. Rev. Lett. **98**, 087204 (2007).
- [10] J. Yu, S. P. Kou and X. G. Wen, arXiv:quant-ph/07092276.
- [11] X. G. Wen, Phys. Rev. D **68**, 065003 (2003).
- [12] For a non-Abelian topological phase, there exists another topological invariable - the Chern number defined by  $v = \frac{1}{2\pi i} \int Tr(U(\mathbf{k})dU(\mathbf{k}) \wedge U(\mathbf{k}))$  with  $U(\mathbf{k}) = \sum_{\alpha} u_{\alpha}(\mathbf{k})M_{\alpha}$  [4, 13]. In our classification, we consider a fixed Chern number.
- [13] N.Read, D.Green, Phys. Rev. B **61**, 10267 (2000).
- [14] D.-H. Lee, G.-M. Zhang, and T. Xiang, arXiv:cond-mat/07053499.